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VARIATIONAL CALCULUS APPLIED TO ELECTRICAL ENGINEERING PROBLEMS

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Abstract- In this chapter theoretical aspects and applications of variational calculus to electrical engineering are presented. At beginning the basics of this type of calculus is presented, for direct problems, and the Euler-Lagrange equation is obtained, considering functionals depending on the first derivative and higher. Then the inverse problems are considered and the form of the functionals associated with the self-adjoint operators that govern the most used Partial Differential Equations (PDEs) in electromagnetics. Then the Rayleigh-Ritz and the weighted residual methods are presented. Application of the Galerkin method to a stationary magnetic field is described and solved symbolically and numerically. Aspects regarding the calculus of variations applied to electro-magnetic field finish the theoretical part. Then two case studies follow. Application of the Ritz method to a 1D electrostatic field problem and applications to electrical circuits.

Keywords: Calculus of Variations, Energy Functional, Rayleigh-Ritz Method, Weighted Residuals Method, Galerkin Method.

1. INTRODUCTION

Variational calculus has widespread applications in science and engineering, such as in analytical mechanics, computational mechanics, differential and computational geometry, optics, electromagnetics, modern physics, fluid mechanics and optimal control. There are many books and articles in the literature regarding different areas, such as [1-5]. They are applied to integer or fractional differential equations problems.

This chapter approaches the applications of variational principles to electrical engineering. Certain foundations relevant for this topic are presented in [6-13].

This type of calculus is referring to the mathematical theories of the extremum principles. When solving problems from mathematical physics and engineering, instead of solving directly differential equations we can look for the minimum value of a definite integral, that generally has an energy significance. This type of problem is called a variational problem and methods that allow us to obtain this approach are called variational methods. The minimum principles have the advantages of being more suggestive than the direct evaluation of the differential Equations. There are certain advantages of the classical approach for continuous problems over the differential formulation regarding the approximate solution [3].

- First, the functional, which may actually represent some physical quantity in the problem, contains lower-order derivatives than the differential operator, and the approximate solution belongs to a larger class of functions.
- Second, if the problem has reciprocal variational formulations, this means, one functional to be minimized and another functional of a different form to be maximized, then certain significant engineering aspects can be found. Another aspect indicates that the variational formulation can be used to consider complicated boundary conditions as natural boundary conditions. Finally, the calculus of variations it can sometimes be used to prove the existence of a solution.

In the past, when engineers used the finite element method to solve their particular continuum problems, they most often relied on calculus of variations formulations to derive the finite element equations. This approach is especially convenient when it is applicable; but before it can be used, a variational statement for the continuum problem must be obtained, that is, we must pose the problem in variational form.

Also, those principles allow obtaining of satisfactory solutions for large classes of engineering problems that have no analytical solutions. The calculus of variations represents the fundamentals for the method of moments (MOM) and the finite element method (FEM) [8-10, 12,28]. In [13] applications of FEM method to magnetic shielding problems are presented.

There are direct and indirect variational methods. The direct method is represented by the Rayleigh-Ritz method. The indirect methods include the method of weighted residuals, Galerkin, Kantorovich, Euler, and the least square methods. A variational principle is applicable to continuous media, thus, ensuring its applicability to the behaviour of fields, as the electromagnetic field.

These methods generate accurate results without having a high computational cost and can be applied to electromagnetic field as well as to electric circuits' problems. Space-time formulations have been used widely in recent years. This approach is applied simultaneously in space and time to Maxwell equations in [14]. In [15] the Rayleigh-Ritz method is used to accelerate transient and/or nonlinear eddy-current analyses.

In the Theory of Electrical Circuits and Electromechanical Systems different methods based on Kirchhoff's laws are known for construction of mathematical model of technical applications. As an engineering analysis tool, the variational approach is well established and has a solid foundation in both physics and mathematics, based on Hamilton variational principle. It says that from all possible movements of conservative mechanical systems in any time interval such movement occurs for which the functional reaches extreme (steady) value. Physically interpreted the method displaces a system from its dynamic equilibrium position and examines the displaced behaviour. Using the calculus of variations, a unique solution for an electrical or magnetic circuit is found via the stationary point of a suitable energy or power expression.

This approach has been described in the literature by examining the behaviour of some linear and nonlinear electric and magnetic circuits. The analysis of capacity circuit is done in a non-traditional way based on the minimum energy principle in [16, 17]. Variational solution of a resistor circuit DC sources [18, 19], is based on the minimum power dissipated. It is also possible to apply the calculus of variations to networks with nonlinear resistors [18, 20], based energy conservation and minimum power dissipated, however, construction of the functional requires evaluation of integrals, in addition to the effort to solve the resulting set of equations. In the case of systems consisting of the so-called higher-order elements, Hamilton's principle is extended to circuits containing the classical resistors and Frequency Dependent Negative Resistors in [22, 23].

The structure of the chapter is as follows: 1. Introduction; 2. Direct problems of variational calculus; 3. Inverse problems of variational calculus; 4. Rayleigh-Ritz method; 5. Weighted Residuals Method; 6. The variational mathematical model of the electromagnetic field; 7. Case study 1: Application of the Ritz method to a 1D electrostatic field problem; 8. Case study 2: Circuit analysis using variational approach; 9. Conclusions.

2. DIRECT PROBLEMS OF VARIATIONAL CALCULUS

The calculus of variation is a chapter from mathematics that is concerned about the finding the extremum of a functional (minimum, maximum or stationary). A functional is a mapping from a function (or a set of functions) to a number, as in Equations (1) and (2). For example, in [1-3, 5] and [9-10]:

$$I[y(x)] = \int_{x1}^{x2} [y(x)]^2 dx \tag{1}$$

$$I[y(x)] = \int_{x1}^{x2} F\left(x, y(x), y'(x)\right) dx \tag{2}$$

where, $y'(x) \equiv \dfrac{dy(x)}{dx}$.

Many engineering problems can be formulated using the calculus of variations, in which we need to find a function $y(x)$ that minimizes (or maximizes) this functional and is subjected to certain essential (boundary) conditions, such as $y(x_1) = a$, and $y(x_2) = b$. For a function $f(x)$, its differential, df, is how much f changes if its argument, x changes by an infinitesimal amount dx. For a functional $I\left[y(x)\right]$, the corresponding term is its (first) variation, δI. δI is how much I changes if its argument, the function $y(x)$, changes by an infinitesimal amount $\delta y(x)$.

This is illustrated in Figure 1, where function $y(x)$ is shown with continuous lie and the varied function first variation is with dotted lines. If the function undergoes a small change $\tilde{y}$ as in Equation (3):

$$\tilde{y}(x) = y(x) + \delta y(x) \tag{3}$$

where, $\delta y(x)$ is a small, continuous function, then the variation of the functional is Equation (4):

$$\delta I = I\left[\tilde{y}(x)\right] - I\left[y(x)\right] \tag{4}$$

We intend to obtain the explicit expression of δI. The operator δ is called the variational symbol. The variation δy of y vanishes at points where y is prescribed, and it is arbitrary elsewhere (Figure 1).

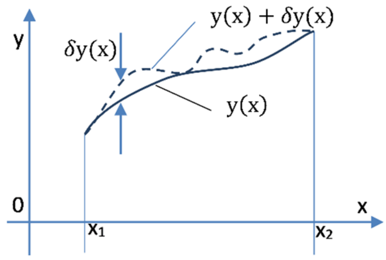

Figure 1. Variation of the extremizing function $y(x)$

Due to the change in y (i.e., $y \to y + \delta y$), there is a corresponding change in I. The first variation of I at y is defined as it can be seen in Equation (5).

$$\delta F = \frac{\delta F}{\delta y}\delta y + \frac{\delta F}{\delta y'}\delta y' \tag{5}$$

This is analogous to the total differential of I, as it can be seen in Equation (6):

$$\delta F = \frac{\delta F}{\delta x}\delta x + \frac{\delta F}{\delta y}\delta y + \frac{\delta F}{\delta y'}\delta y' \tag{6}$$

where, $\delta x = 0$ since x does not change as y changes to $y + \delta y$. Thus, we note that the δ operator is similar to the differential operator. Therefore, if $F_1 = F_1(y)$ and $F_2 = F_2(y)$ [1-3, 5] then the corresponding mathematical operations are as in Equations (7) to (12):

$$\delta(F_1 \pm F_2) = \delta F_1 \pm \delta F_2 \tag{7}$$

$$\delta(F_1 F_2) = F_2\, \delta F_1 \pm F_1 \delta F_2 \tag{8}$$

$$\delta\left(\frac{F_1}{F_2}\right) = \frac{F_2\, \delta F_1 \pm F_1 \delta F_2}{F_2^2} \tag{9}$$

$$\delta(F_1)^n = n(F_1)^{n-1}\delta F_1 \tag{10}$$

$$\frac{d}{dx}\delta y = \delta\left(\frac{dy}{dx}\right) \tag{11}$$

$$\delta\int_a^b y(x)dx = \int_a^b \delta y(x)dx \tag{12}$$

A necessary condition for the function $I(y)$ in Equation (2) to have an extremum is that the variation vanishes, as in Equation (13).

$$\delta I = 0 \tag{13}$$

To apply this condition, we must be able to find the variation δI of I, as in Equation (4). Let $h(x)$ be an increment in $y(x)$. The boundary conditions are satisfied by $y(x)$ + $h(x)$ if Equation (14) is true:

$$h(x_1) = h(x_2) = 0 \tag{14}$$

The corresponding increment in $I(y)$ in Equation (2) is [2, 3, 9, 13]:

$$\Delta I = I(y+h) - I(y) = \\ = \int_{x1}^{x2}[F(x, y+h, y'+h') - F(x, y, y')]dx \tag{15}$$

Applying Taylor's expansion, we obtain Equation (16):

$$\Delta I = \int_a^b [F_y(x,y,y')h - F_{y'}(x,y,y')h']dx = \delta I + O(h^2) \tag{16}$$

where, $\delta I = \int_a^b [F_y(x,y,y') - F_{y'}(x,y,y')h']dx$.

Integrating by parts and imposing $\delta I = 0$ the following Equation (17) is obtained [2, 3, 5]:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{17}$$

This is called Euler's (Euler-Lagrange) equation. Thus, a necessary condition for $I(y)$ to have an extremum for a given function $y(x)$ is that $y(x)$ must satisfy Euler's equation. This idea is extended to more general cases in Equation (18), for example when the functional depends on the second and higher order derivatives [1, 2, 3, 10]:

$$I(y) = \int_a^b F(x, y, y', y'', ..., y^{(n)})dx \tag{18}$$

In this case the corresponding Euler's Equation becomes Equation (19):

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} - \frac{d^3}{dx^3}F_{y'''} + ... + \\ + (-1)^n \frac{d^n}{dx^n}F_{y^{(n)}} = 0 \tag{19}$$

3. INVERSE PROBLEMS OF VARIATIONAL CALCULUS

The Euler's equation produces the differential equation corresponding to a given functional or to a variational principle. The inverse procedure of constructing a variational principle for a given differential equation is of great interest. We consider a differential Equation (20), described by an operator L [3, 10]:

$$Ly = g \tag{20}$$

If L is real, self-adjoint, and positive definite, the functional associated with the previous equation, that would be minimized by the solution of Equation (20) is the functional:

$$I(y) = \langle Ly, y\rangle - 2\langle y, g\rangle \tag{21}$$

where, $< f, g >$ is the scalar or the dot product of functions f and g.

$$< f, g > = \int_\Omega fg^* d\Omega \tag{22}$$

This approach can be applied also to derive solutions of integral equations. Other systematic approaches for the derivation of variational principles of electromagnetic problems include Hamilton's principle or the principle of least action and Lagrange multipliers [1, 2, 3, 6].

The expressions of certain variational functionals corresponding to some of the mostly used PDEs from electromagnetics (wave, diffusion, and Poisson types) are described in [7, 10]. The nonhomogeneous wave is described by Equation (23):

$$\nabla^2\Phi + k^2\Phi = g \tag{23}$$

and the corresponding functional is described by Equation (24):

$$I(\Phi) = \frac{1}{2}\int_v \left[|\nabla\Phi|^2 + k^2\Phi^2 + 2g\Phi\right]dv \tag{24}$$

The diffusion equation is described by Equation (25):

$$\nabla^2\Phi - k^2\Phi_t = 0 \tag{25}$$

and the corresponding functional is described by described by Equation (26):

$$I(\Phi) = \frac{1}{2}\int_v \left[|\nabla\Phi|^2 - \frac{1}{u^2}\Phi_t^2\right]dvdt \tag{26}$$

The Poisson equation is in Equation (27):

$$\nabla^2\Phi = g \tag{27}$$

and the functional has the expression from Equation (28)

$$I(\Phi) = \frac{1}{2}\int_v \left[|\nabla\Phi|^2 - 2g\Phi\right]dv \tag{28}$$

where,

$$|\nabla\Phi|^2 = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \tag{29}$$

For an electromagnetic field the functional has the general form [26, 27]:

$$I = \int_0^t\int_V [\overline{A}\overline{J} - \rho\varphi + \frac{\varepsilon}{2}(-grad\varphi - \frac{\partial\overline{A}}{\partial t})^2 - \frac{1}{2\mu}rot^2\overline{A}]dVd\tau \tag{30}$$

or :

$$I = \int_0^t\int_V \frac{1}{2}[\sigma\,\overline{A}(-grad\varphi - \frac{\partial\overline{A}}{\partial t}) + \varepsilon(-grad\varphi - \frac{\partial\overline{A}}{\partial t})^2 \\ -\frac{1}{\mu}rot^2\overline{A}]dVd\tau \tag{31}$$

4. RAYLEIGH-RITZ METHOD

One general method for obtaining approximate solutions to problems expressed using the calculus of variations form is known as the Ritz method. The Rayleigh-Ritz method it is one of the most important direct variational methods, used both obtaining solutions to problems in physics and engineering, theoretically and practically [3, 7, 9, 10].

Suppose that a function $y(x)$ is sought that must perform the extreme of the functional $I(y)$. The function can be approximated by a linear combination of conveniently chosen independent linear coordinate functions of the form:

$$y(x) \approx \varphi_0(x) + c_1\varphi_1(x) + c_2\varphi_2(x) + ... + +c_n\varphi_n(x) \quad (32)$$

where, the constant coefficients c_n are to be determined.

The functions $\varphi_0(x)$, $\varphi_1(x)$ ... $\varphi_n(x)$ are chosen in advance, so that this expression satisfies the boundary conditions given for any choice of constants $c_1, c_2, ...c_n$. For problems of physical significance, the general nature of the desired solution is often known, and the system of functions is chosen so that a linear combination is expected to represent well enough the solution. Replacing (32) in the functional and performing the integration we obtain:

$$I = I(c_1, c_2, ..., c_n) \quad (33)$$

The value of the functional is now a function of the n unknowns' coefficients: $c_1, c_2, ..., c_n$. In this way the problem of variational calculus is replaced by a common problem of maximum and minimum. The solution is generally obtained by solving a system of n equations with n variables.

$$\frac{\partial I}{\partial c_i} = 0,\ i = 1, 2, ..., n \quad (34)$$

The efficiency of the process depends obviously on choosing convenient approximation functions $\varphi_i(x)$. In practice, the procedure consists in obtaining a succession of approximations where $\varphi_0 + c_1\varphi_1$ is the first, $\varphi_0 + c_1\varphi_1 + c_2\varphi_2$ is the second, $\varphi_0 + c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3$, is the third and generally (32) is the nth approximation.

The values of the constants c must be determined at each stage of the process. However, an important feature of this method is the following one.

If $y_n(x) = \varphi_0(x) + c_1\varphi_1(x) + c_2\varphi_2(x) + ... + c_n\varphi_n(x)$ represents the nth approximation of the true solution $y(x)$, then:

$$y_{n+1}(x) = \varphi_0(x) + c_1\varphi_1(x) + \varphi_2(x) + ... + c_n\varphi_n(x) + \\ +c_{n+1}\varphi_{n+1}(x) \quad (35)$$

the $(n+1)$th approximation, will be better than y_n.

In general, the constants c_k will differ from the corresponding values c_k, $k = \overline{1,n}$. By comparing the successive approximations, an evaluation of the order of accuracy achieved in each calculation step can be determined. We define the convergence of the process as, for $n \to \infty$ the functions $\varphi_0 + \sum_{k=1}^{n} C_n\varphi_n(X)$ converge to the desired function $y(x)$.

In many cases, a complete system of functions is chosen, such as polynomials, sins and cosines, Bessel functions, the choice depending on the shape of the domain, the type of coordinate system, etc. The previous discussion of the Rayleigh-Ritz process can be extended in several ways. Functionals and coordinate functions can contain several independent variables. Thus, suppose we want to find out the minimum of functional:

$$I(u) = \iint_A F(x, y, u, u_x, u_y)dxdy \quad (36)$$

with the limit condition:

$$u = g(s) \quad (37)$$

On the contour C, where $g(s)$ is a position function prescribed on C, and s is the length of the arc, measured from a fixed point on C. Then the approximate solution can be written as:

$$u_n(x, y) = \varphi_o(x, y) + c_1\varphi_1(x, y) + c_2\varphi_2(x, y) + ... + \\ +c_n\varphi_n(x, y) \quad (38)$$

where, φ_0 satisfies (37), $\varphi_k(x, y) = 0$ on C for $k = 1, 2, ..., n$ and $c_1, c_2, ..., c_n$ are constants to be determined so that $I(u_n)$ is extremized.

The coordinate functions are chosen in advance so as to satisfy the boundary conditions. Similarly, our considerations can be extended to a functional which depend on several independent variables, as well as on higher order derivatives appearing in the functional form of the integrand F.

The Rayleigh-Ritz method has two drawbacks [3, 7, 10]. For certain problems described by non-self-adjoint PDEs (odd order derivatives) the variational principle may not exist. Another aspect is that is very difficult to find the functions that satisfies the global boundary conditions for the domains with complicated geometries. For the more general Petrov-Galerkin method, the weighting functions are different from the basic functions.

5. WEIGHTED RESIDUALS METHOD

5.1. Introduction

The Rayleigh-Ritz method can be applied when a suitable functional exists. When such functional is difficult or impossible to find, we apply one of the techniques usually referred to as the method of weighted residuals. The applicability of the method it is not limited to the classes of variational problems.

We consider the equation:

$$Ly = g \quad (39)$$

where, L is a differential or integral operator.

The solution to Equation (39) is approximated, using the expansion functions, φ_n as follows:

$$\tilde{y} = \sum_{1}^{N} c_n\varphi_n \quad (40)$$

where, c_n are the unknowns' coefficients.

Replacing the approximate solution in Equation (40) a residual R is generated:

$$R = L(\tilde{y} - y) \approx L\tilde{y} - g \quad (41)$$

The weighting functions w_m (usually they are not the same as w_n) are chosen such that the integral of a weighted residual is zero, in some sense:

$$\int w_m R dv = 0 \quad (42)$$

or using the scalar product:

$$\langle w_m, R \rangle = 0 \quad (43)$$

If a set of weighting functions w_m (also known as testing functions) are chosen and the inner product is considered for each w_m, we obtain Equation (44):

$$\sum_1^N c_n \langle w_m, L\varphi_n \rangle = \langle w_m, g \rangle \quad (44)$$

where, $m = \overline{1, N}$ .The matrix form of system of linear equations (44) is:

$$[A][C] = [G] \quad (45)$$

where, $A_{mn} = \langle w_m, L\varphi_n \rangle$, $G_m = \langle w_m, g \rangle$, $C_n = c_n$. Solving system (45) and replacing the obtained c_n coefficients in Equation (40) we found the approximate solution to Equation (39).

There are different ways of choosing the weighting functions w_n that generate different methods:

- collocation (or point-matching method),
- subdomain method,
- Galerkin method,
- least squares method.

If the operator L is a linear differential operator of even order and we consider the weighting functions being the same with the basic functions, i.e., $w_m = \varphi_m$ then the Galerkin method reduces to the Rayleigh-Ritz method. This is due to the fact that the differentiation can be transferred to the weighting functions and the resulting coefficient matrix [A] will be symmetric [3, 10].

5.2. Method of Weighted Residuals for a Poisson Partial Differential Equation

We consider a Poisson PDE:

$$\nabla \cdot k \nabla u + g = 0 \quad (46)$$

where, u is a potential (electric or magnetic) defined in a domain D with a boundary Γ, with Dirichlet or Neumann boundary conditions. In order to obtain the numerical solution of Equation (46) we can use Equation (40) and express the potential u [7, 9, 24]:

$$u = \sum N_i a_i \quad (47)$$

where, N_i are suitable chosen shape functions or trial functions and a_i are parameters (potential values) that should be found. Replacing Equation (47) into Equation (46) we obtain a certain residue:

$$R = \left(\nabla k \nabla \sum N_i a_i + g\right) \neq 0 \quad (48)$$

where, in general R vanishes if a_i is the exact solution.

Equation (48) is the residue equation, and it measures the error introduced by using the approximate solution Equation (47). The Equation (42) is known as the weighted residual approximation of the solution. If the number of weighting functions is chosen equal to the number of trial parameters a system of linear equations is obtained:

$$\sum \left(\int_\Omega w_i \nabla \cdot k \nabla N_j d\Omega \right) a_i = -\int_\Omega w_i g d\Omega \quad (49)$$

which is of the form:

$$K_{ij} a_j = g_j \quad (50)$$

This formulation allows the unification of the commonly used methods as Equation (39); specific techniques are available for Equation (49) by particular choices of the weighting function. For exemplification, we consider the problem of finding the magnetic field inside a highly permeable rectangular conductor.

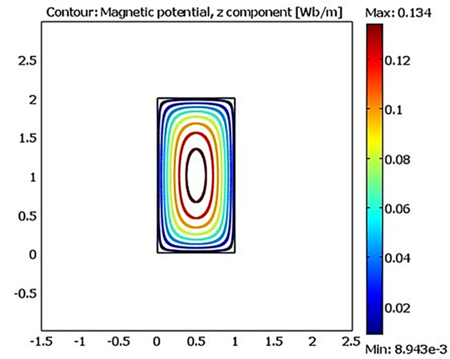

Figure 2. Magnetic potential Az contour lines

This example satisfies the linear Poisson equation, so that:

$$\nabla^2 A_z = -\mu J_z = -g \quad (51)$$

subjected to the following boundary conditions:

$$B_n = \begin{cases} \dfrac{\partial A}{\partial y} = 0 & x = \pm a \\ \dfrac{\partial A}{\partial x} = 0 & y = \pm b \end{cases} \quad (52)$$

- Example 1 [24]:

Considering a very simple trial function with just one unknown parameter A_1 we obtain:

$$A = (x^2 - a^2)(y^2 - b^2)A_1 = N_1 A_1 \quad (53)$$

That verifies the boundary conditions Equation (52) Applying Galerkin weighting

$$w_j = N_j \quad (54)$$

Equation (49) becomes:

$$\left(\int N_1 \nabla^2 N_1 dxdy \right) A_1 = -\mu J \int_\Omega N_1 dxdy \quad (55)$$

Replacing N_1 we obtain:

$$\begin{aligned} &\left(2\int (x^2 - a^2)(y^2 - b^2)\nabla^2 \left[(x^2 - a^2)(y^2 - b^2) \right] dxdy \right) A_1 = \\ &= -\mu J \int_\Omega (x^2 - a^2)(y^2 - b^2) dxdy \end{aligned} \quad (56)$$

After mathematical manipulations, the value of A is found:

$$A = \frac{5}{8}\mu\mu_0 J \frac{\left[(x^2-a^2)(y^2-b^2)\right]}{a^2+b^2} \tag{57}$$

• Example 2[45]:

In this example consider a two-parameter trial function is considered:

$$A = (x^2-a^2)(y^2-b^2)A_1 + x^2(x^2-a^2)(y^2-b^2)A_2 \tag{58}$$

or:

$$A = N_1 A_1 + N_2 A_2 \tag{59}$$

Applying Galerkin weighting to Equation (56) a system of algebraic equations results:

$$k_{11}A_1 + k_{12}A_2 = f_1 \tag{60}$$

$$k_{21}A_1 + k_{22}A_2 = f_2 \tag{61}$$

where,

$$k_{ij} = \int N_1 \nabla^2 N_1 dxdy \tag{62}$$

$$g_i = -\mu J \int N_i dxdy \tag{63}$$

Replacing N_i in Equations (61) and (62) and performing the integration, the following system results:

$$\begin{bmatrix} a^2+b^2 & a^2\left(\frac{a^2}{7}+\frac{b^2}{5}\right) \\ \left(\frac{a^2}{7}+\frac{b^2}{5}\right) & \frac{a^2}{7}\left(\frac{a^2}{3}+\frac{11b^2}{5}\right) \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \frac{\mu J}{8} \tag{64}$$

The solution is:

$$A = \begin{pmatrix} \frac{175\mu J(5a^2+33b^2)}{32(25a^4+280a^2b^2+252b^4)} - \frac{105J(5a^2+7b^2)}{32(25a^4+280a^2b^2+252b^4)} \\ \frac{3675\mu J(a^2+b^2)}{32(25a^6+280a^4b^2+252a^2b^4)} - \frac{525J(5a^2+7b^2)}{32(25a^6+280a^4b^2+252a^2b^4)} \end{pmatrix} \tag{65}$$

In Figure 3 the values of the magnetic vector potential are compared, along the median line of the conductor, from (-a,0) to (a,0), considering the results from finite element method software Comsol Multiphysics and those from the Galerkin approximation considering relation (53).

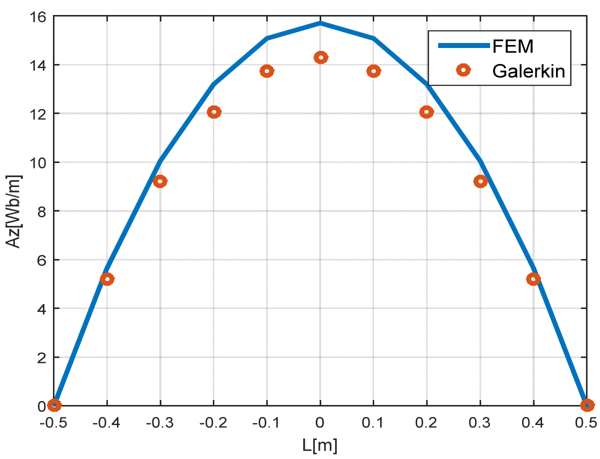

Figure 3. Comparison between Galerkin and FEM values of the magnetic vector potential A

It can be noticed that even for the one parameter approximation Galerkin method generates reasonable results. The error increases as we get closer to the centre of the conductor.

6. THE VARIATIONAL MATHEMATICAL MODEL OF THE ELECTROMAGNETIC FIELD

The analysis of the macroscopic electromagnetic field admits besides the differential formulation also an equivalent variational formulation [6, 7, 8, 11, 12]. The construction of that formulation assumes the setting of a variational principle (of Lagrangian or Hamiltonian type) capable of offering from the stationary condition of an adequate functional, the electromagnetic field in bodies.

Certain aspects regarding the formalism of Lagrangian type associated to the electromagnetic potentials will be described. Both physical and intuitive significance of the natural energy functionals are emphasized. If x_k are the independent variables (including time), that describes a physical system, y_j the dependent variables and y_{jk} the partial derivatives of first order:

$$y_{jk} = \partial y_j / \partial x_k \tag{66}$$

The variational principle of the stationary action postulates the existence of an integral type of functional S, called action, of the following form:

$$S = \int_{\Omega} L(x_k, y_j, y_{jk}) d\Omega \tag{67}$$

The action S poses a stationary value (or an extreme value) corresponding to the real evolution of the considered system. The integrand L is called Lagrangian and represents a scalar function of the respective physical system. The Lagrangian signifies the difference between two terms: one of kinetic co energy density type and the other of potential energy density type:

$$L = w_c^* - w_p \tag{68}$$

The necessary stationary condition, Equation (13), consists of cancelling its first variation:

$$\delta S = \delta \int_{\Omega} L(x_k, y_j, y_{jk}) d\Omega = 0 \tag{69}$$

Generally, the action S must contain three additive terms [6, 7, 8, 9, 11]:

– The first term depends on the properties of the material bodies in the absence of the field. It is omitted because it does not intervene in the computation of the electromagnetic field.

- The second term describes the free electromagnetic field

- The last term describes the electromagnetic interaction between field and bodies.

As a consequence, the action integral is considered as a functional of energetic type of the following form:

$$S = S_C + S_{CC} = \int_D \left(L_c + L_{cc}\right) dD =$$
$$= \int_D \left\{ \left(\int_0^{\overline{E}} \overline{D} d\overline{E} - \int_0^{\overline{B}} \overline{H} d\overline{B} \right) + \left(\overline{J}\,\overline{A} - \rho_v V \right) \right\} dD \tag{70}$$

where, the expressions of the Lagrangians L_c and L_{cc} were emphasized.

The expression of L_c is justified considering that:

- It represents a scalar function of the electric and magnetic field components.

- It is in concordance with the definition (81)

The Lagrangian $L_{cc} = \overline{J}\,\overline{A} - \rho_v V$ defines the difference between the volume densities of the magnetic field and electric field, respectively. The Maxwell equations for quasi stationary regime are the following:

$$curl\overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{71}$$

$$curl\overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \tag{72}$$

$$div\overline{J} = -\frac{\partial \rho_V}{\partial t} \tag{73}$$

$$div\overline{D} = \rho_V \tag{74}$$

$$div\overline{B} = 0 \tag{75}$$

The electric and magnetic potential V and A are introduced using the following relations:

$$\overline{B} = curl\overline{A} \tag{76}$$

$$\overline{E} = -gradV - \frac{\partial \overline{A}}{\partial t} \tag{77}$$

The magnetic flux density $\overline{B}$ and the electric field $\overline{E}$ satisfy Equation (74) and Equation (75), being uniquely determined by Equation (76) and Equation (77). It can be proven [8] that for arbitrary and independent variations of the potential functions $\overline{A}$ and V, applying (69) to functional (70) we can be obtain the Maxwell equations. If the first variation of the functional (70) is considered, then we obtain:

$$\begin{aligned}
&\delta S(\delta\overline{A},\delta V) = \int_D \left\{ \left(\int_{\overline{0}}^{\overline{E}+\delta\overline{E}} \overline{D}d\overline{E} - \int_{\overline{0}}^{\overline{B}+\delta\overline{B}} \overline{H}d\overline{B} \right) + \left(\overline{J}\delta\overline{A} - \rho_v \delta V \right) \right\} dD = \\
&\int_D \left\{ \left(\overline{D}d\overline{E} - \overline{H}d\overline{B} \right) + \left(\overline{J}\delta\overline{A} - \rho_v \delta V \right) \right\} dD = \\
&\int_D \left\{ \overline{D}\left(-\frac{\partial\left(\delta\overline{A}\right)}{\delta t} - \overline{H}d\overline{B} \right) + \overline{H}rot\delta\overline{A} + \overline{J}\delta\overline{A} - \rho_v \delta V \right\} dD = \\
&\int_D \left\{ \left(\overline{J} + \frac{\partial\left(\overline{D}\right)}{\delta t} - \overline{H}d\overline{B} - rot\overline{H} \right) + \delta\overline{A} + \left(div\overline{D} - \rho_v \right)\delta V \right\} dD + \\
&\int \left\{ -\frac{\partial}{\delta t}\left(\overline{D}\delta\overline{A} \right) - div\left(\overline{D}\delta\overline{V} \right) - div\left(\overline{A}\times\overline{H} \right) \right\} dD = 0
\end{aligned} \tag{78}$$

Applying the Gauss theorem to the terms that includes divergence, in the second integral and considering the boundary of the domain D at infinite then the last two terms vanish. Also, the first term is zero because $\delta\overline{A} = 0$ at the ends of the temporal range, after the time integration.

As a consequence, the first integral is zero and the equations (72) and (74) are fulfilled. Equation (73) can be deduced from (72) and (74) as a dependent equation. The variational handling of a concrete analysis problem of the electromagnetic field implies [8]:

- Customizing the action integral (70) according to the electromagnetic field regime and to the physical state of the field media
- The inclusion into the energetic functional of the uniqueness conditions
- Solving the problem to find the potential function that cancels the first variation of action.

7. CASE STUDY 1: APPLICATION OF THE RITZ METHOD TO A 1D ELECTROSTATIC FIELD PROBLEM

The electrostatic field equations are:

$$\nabla \times \overline{E} = 0 \tag{79}$$

$$\nabla\overline{D} = \nabla(\varepsilon\overline{E}) = \rho_v \tag{80}$$

where, $\overline{E}$ is the electrostatic field, $\overline{D}$ is the electric induction, ε is the electric permeability of the media and ρ_v is the volume charge density.

Equation (79) indicates that $\overline{E}$ can be written as the gradient of a scalar field V, the electrostatic potential:

$$\overline{E} = -\nabla V \tag{81}$$

Replacing (79) in (80) we obtain:

$$\nabla(\varepsilon(-\nabla V)) = -\varepsilon\nabla^2 V = \rho_v \tag{82}$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \tag{83}$$

The problem is to solve the Poisson equation (83) and determine the potential V that verifies certain boundary conditions. We consider two infinite (extended in Oy direction) conducting plates placed at x=0 and x=a, respectively as in Figure 4.

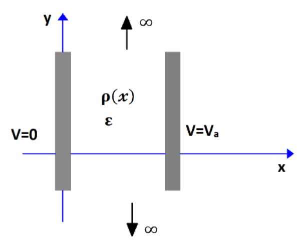

Figure 4. 1D electrostatic problem

7.1. Differential Formulation

Let us consider that the change is distributed with a charge density, $\rho_v = \rho(x) = \rho_0 \cdot x$ between plates [25]. Because the potential V depends only on x coordinate Equation (83) becomes:

$$\frac{d^2V}{dx^2} = -\frac{\rho_0}{\varepsilon_0}x \tag{84}$$

Considering $\frac{\rho_0}{\varepsilon_0} = 1$, Equation (84) becomes:

$$\frac{d^2V}{dx^2} = -x \tag{85}$$

Integrating twice Equation (85) we obtain the analytical solution:

$$V(x) = -\frac{x^2}{6} + c_1 x + c_2 \tag{86}$$

Using the boundary conditions $V(0)=0$ and $V(a)=V_a$, c_1, c_2 are obtained, and the solution becomes:

$$V(x)=-\frac{x^2}{6}+\frac{a^2}{6}x+\frac{v_a}{a}x \quad (87)$$

We will consider $a=1m$ and $V_a=100V$ .We would like to approximate the solution using the Ritz method. The potential V will be approximated with $\tilde{V}$ considering the following expression:

$$\tilde{V}(x)=C(x-x^2)+100x \quad (88)$$

Electric potential $\tilde{V}(x)$ satisfies the boundary conditions: $\tilde{V}(0)=0$ and $\tilde{V}(1)=100$. The functional that will be minimized has the expression:

$$I(V)=\frac{1}{2}\int_V (|\nabla V|^2-2xV)dx \quad (89)$$

Replacing (88) in (89) and performing the integration we obtain for the functional the expression:

$$I(V)=-\frac{c^2}{3}+10000-\frac{2c}{3}+\frac{c}{2}-\frac{200}{3} \quad (90)$$

Minimizing the functional $I(V)$ we obtain the value of the c parameter from (90):

$$\frac{\partial I}{\partial c}=0 \quad (91)$$

This gives $c=\frac{1}{4}$. Because the second derivative is positive: $\frac{\partial^2 I}{\partial^2 c}=\frac{2}{3}>0$ it confirms the minimum of the functional $I(V)$.

The approximation $\tilde{V}(x)$ has the final expression:

$$\tilde{V}(x)=\frac{1}{4}\left(x-x^2\right)+100x \quad (92)$$

In Figure 5. the exact solution $V(x)$ and $\tilde{V}(x)$ are compared and a good agreement is noticed.

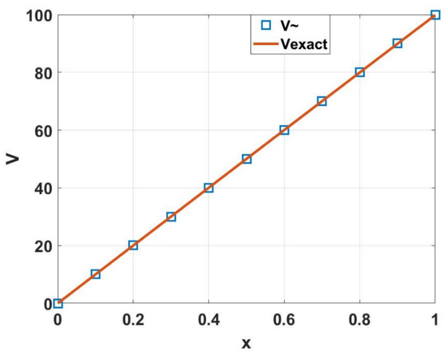

Figure 5. Comparisons between the analytical and Ritz solution

The results from Figure 5 show a good agreement between the analytical solution (87) and the Ritz approximation (92). Generally, in order to improve the precision more terms should be considered in the approximation.

8. CASE STUDY 2: CIRCUIT ANALYSIS USING VARIATIONAL APPROACH

8.1. Solution of Capacitor Circuits

The analysis of a capacity circuits is done in a traditional way, based on Kirchhoff's laws, and in a non-traditional way, based on the variational approach (minimum energy principle).

8.1.1. Application 1

Consider the capacitor circuit illustrated in Figure 6. For the numerical values of parameters of branch elements: $E_1=20\text{V}$, $E_2=40\text{V}$, $C_1=2\mu\text{F}$, $C_2=4\mu\text{F}$, $C_3=5\mu\text{F}$, $C_4=20\mu\text{F}$, $C_5=10\mu\text{F}$, it is required to determine the voltages at the capacitor terminals.

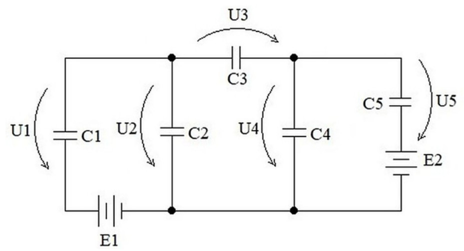

Figure 6. Capacitor circuit with DC sources

Applying the solving algorithm with Kirchhoff's theorems, for static electrical circuits, we obtain the following linear system of equations, Equations (93), (94):

$$\begin{bmatrix} C_1 & C_2 & C_3 & 0 & 0 \\ 0 & 0 & -C_3 & C_4 & C_5 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_1 \\ 0 \\ E_2 \end{bmatrix} \quad (93)$$

$$\begin{bmatrix} 2\times10^{-6} & 2\times10^{-6} & 5\times10^{-6} & 0 & 0 \\ 0 & 0 & -5\times10^{-6} & 20\times10^{-6} & 10\times10^{-6} \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \\ 0 \\ 40 \end{bmatrix} \quad (94)$$

with solutions:

$U_1=-10.56\text{V}, U_2=9.44\text{V}, U_3=-3.33\text{V}, U_4=12.77\text{V}, U_5=-27.23\text{V}$.

Next, an alternative solution method based on the Hamiltonian variational (energetic) principle [16, 17] will be used. Accordingly, the voltage on the sides of the circuit is distributed so that the energy of the electric field is minimal. Considering the potentials of the n nodes of the circuit as unknowns, up to n-1, because the potential of a node is considered the reference, according to Figure 7, results:

$U_1=V_1-E_1=V_1-20, U_2=V_2, U_3=V_1-V_2, U_4=V_2$, $U_5=V_2-E_2=V_2-40$.

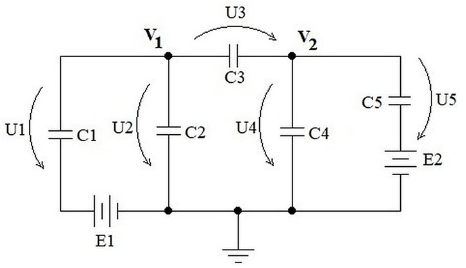

Figure 7. Capacitor circuit with DC sources and unknowns' voltages V_1 and V_2

The functional based of the electrical energy of the system is described in Equation (95):

$$W_{elst}(V_1,V_2)=\frac{1}{2}2\times10^{-6}(V_1-20)^2+\frac{1}{2}4\times10^{-6}V_1^2+$$
$$+\frac{1}{2}5\times10^{-6}(V_1-V_2)^2+\frac{1}{2}20\times10^{-6}V_2^2+\frac{1}{2}10\times10^{-6}(V_2-40)^2 \quad (95)$$

To find the stationary point, minimum of functional $W_{elst}(V_1,V_2)$, differentiate $W_{elst}(V_1,V_2)$ with respect to V_1 and then V_2, and set each result to zero as in Equations (96), (97), (98).

$$\frac{\partial W_{elst}(V_1,V_2)}{\partial V_1}=2\times10^{-6}(V_1-20)+4\times10^{-6}V_1+$$
$$+5\times10^{-6}(V_1-V_2)=0 \quad (96)$$

$$\frac{\partial W_{elst}(V_1,V_2)}{\partial V_2}=-5\times10^{-6}(V_1-V_2)+20\times10^{-6}V_2+$$
$$+10\times10^{-6}(V_2-40)=0 \quad (97)$$

$$\begin{bmatrix}11 & -5\\ -5 & 35\end{bmatrix}\cdot\begin{bmatrix}V_1\\ V_2\end{bmatrix}=\begin{bmatrix}40\\ 400\end{bmatrix} \quad (98)$$

The obtained results are: $V_1=9.44V, V_2=12.77V$.

The nature of the turning point is obtained forming the second order derivatives of functional $W_{elst}(V_1,V_2)$, Equations (99).

$$\frac{\partial^2 W_{elst}(V_1,V_2)}{\partial V_1^2}=2\times10^{-6}+4\times10^{-6}V_1+5\times10^{-6}=11\times10^{-6}>0$$
$$\frac{\partial^2 W_{elst}(V_1,V_2)}{\partial V_2^2}=5\times10^{-6}+20\times10^{-6}+10\times10^{-6}=35\times10^{-6}>0 \quad (99)$$

Because both are positive quantities, the turning point of electrostatic energy is a minimum, $W_{elst}(V_1,V_2)_{\min}=5.655$ mJ.

The voltage V_1 and V_2 are treated as trial functions and varied up to 20 and 40 volts respectively, then $W_{elst}(V_1,V_2)$ exhibits a minimum value of 5.7mJ for $V_1=9.4445\text{V}, V_2=12.7778\text{V}$. The minimum value, where obtained using the Nelder-Mead algorithm, implemented in MATLAB, using the function fminsearch, with 85 iterations, Figure 8, It is a multidimensional unconstrained nonlinear minimization technique.

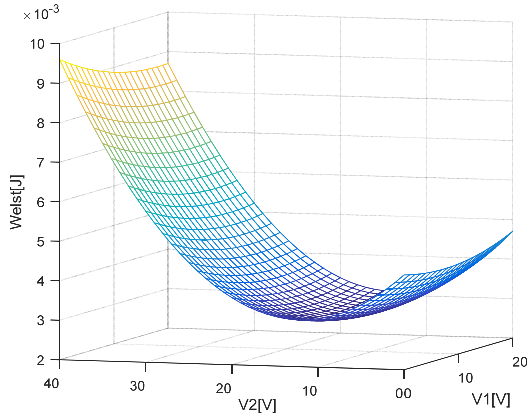

Figure 8. The distribution of the electrostatic energy versus V_1 and V_2

8.1.2. Application 2

In this application for capacitor circuits the advantage of the variational method on more complex networks will be highlighted. For the capacitor circuit illustrated in Figure 9, the numerical values of parameters of branch elements are:

$E_1=50\text{V}, E_2=200\text{V}, C_1=1\mu\text{F}, C_2=2\mu\text{F}, C_3=3\mu\text{F},$
$C_4=4\mu\text{F}, C_5=5\mu\text{F}, C_6=6\mu\text{F}, C_7=7\mu\text{F}, C_8=8\mu\text{F}.$

The request is to determine the voltages at the capacitor terminals.

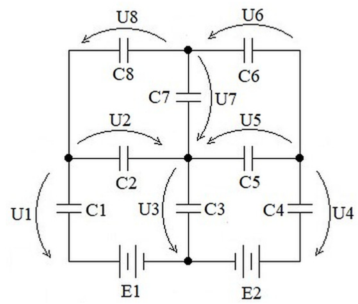

Figure 9. Capacitor circuit with DC sources

The solving with Kirchhoff's theorems, we obtain the following linear system of equations, as Equations (100), (101):

$$\begin{bmatrix}C_1 & C_2 & 0 & 0 & 0 & 0 & 0 & C_8\\ 0 & -C_2 & C_3 & 0 & C_5 & 0 & C_7 & 0\\ 0 & 0 & 0 & 0 & 0 & -C_6 & C_7 & C_8\\ 0 & 0 & 0 & C_4 & C_5 & C_6 & 0 & 0\\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0\\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0\end{bmatrix}\cdot\begin{bmatrix}U_1\\ U_2\\ U_3\\ U_4\\ U_5\\ U_6\\ U_7\\ U_8\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\\ E_1\\ 0\\ 0\\ E_2\end{bmatrix} \quad (100)$$

$$\begin{bmatrix} 1\times10^{-6} & 2\times10^{-6} & 0 & 0 & 0 & 0 & 0 & 8\times10^{-6} \\ 0 & -2\times10^{-6} & 3\times10^{-6} & 0 & 5\times10^{-6} & 0 & 7\times10^{-6} & 0 \\ 0 & 0 & 0 & 0 & 0 & -6\times10^{-6} & 7\times10^{-6} & 8\times10^{-6} \\ 0 & 0 & 0 & 4\times10^{-6} & 5\times10^{-6} & 6\times10^{-6} & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \\ 200 \end{bmatrix} \quad (101)$$

with solutions:
$U_1 = 43.45\text{V}, U_2 = 5.08\text{V}, U_3 = 88.37\text{V}, U_4 = -77.15\text{V},$
$U_5 = 34.48\text{V}, U_6 = 22.69\text{V}, U_7 = 11.79\text{V}, U_8 = 53.68\text{V}.$

Applying the variational approach, similarly with Application 1, the potentials of the n-1 nodes of the circuit will be determined, the potential of the nth node being the reference. In accord with Figure 10, results:
$U_1 = V_1 - E_1 = V_1 - 50, U_2 = V_1 - V_2, U_3 = V_2,$
$U_4 = V_4 - E_2 = V_4 - 200, U_5 = V_4 - V_2, U_6 = V_4 - V_3,$
$U_7 = V_3 - V_2, U_8 = V_3 - V_1.$

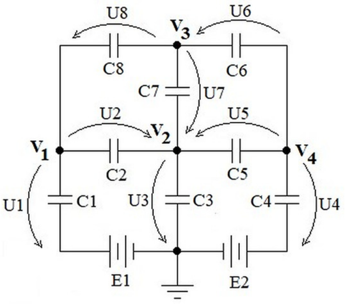

Figure 10. Capacitor circuit with DC sources and unknowns' voltages V_1, V_2, V_3 and V_4

The functional based of the electrical energy of the system is described by Equation (102):

$$W_{elst}(V_1,V_2,V_3,V_4) = \frac{1}{2}1\times10^{-6}(V_1-50)^2 + \frac{1}{2}2\times10^{-6}(V_1-V_2)^2 + \\ + \frac{1}{2}3\times10^{-6}V_2^2 + \frac{1}{2}4\times10^{-6}(V_4-200)^2 + \frac{1}{2}5\times10^{-6}(V_4-V_2)^2 + \\ + \frac{1}{2}6\times10^{-6}(V_4-V_3)^2 + \frac{1}{2}7\times10^{-6}(V_3-V_2)^2 + \frac{1}{2}8\times10^{-6}(V_3-V_1)^2 \quad (102)$$

Differentiate $W_{elst} = (V_1, V_2, V_3, V_4)$ with respect to V_1, V_2, V_3 and V_4, and set each result to zero, we reach to Equations (103) and (104):

$$\frac{\partial W_{elst}(V_1,V_2,V_3,V_4)}{\partial V_1} = 10^{-6}(V_1-50) + \\ +2\times10^{-6}(V_1-V_2) - 8\times10^{-6}(V_3-V_1) = 0 \\ \frac{\partial W_{elst}(V_1,V_2,V_3,V_4)}{\partial V_2} = -2\times10^{-6}(V_1-V_2) + \\ +3\times10^{-6}V_2 - 5\times10^{-6}(V_4-V_2) - 7\times10^{-6}(V_3-V_2) = 0 \\ \frac{\partial W_{elst}(V_1,V_2,V_3,V_4)}{\partial V_3} = -6\times10^{-6}(V_4-V_3) + \\ +7\times10^{-6}(V_3-V_2) + 8\times10^{-6}(V_3-V_1) = 0 \\ \frac{\partial W_{elst}(V_1,V_2,V_3,V_4)}{\partial V_4} = 4\times10^{-6}(V_4-200) + \\ +5\times10^{-6}(V_4-V_2) + 6\times10^{-6}(V_4-V_3) = 0 \quad (103)$$

$$\begin{bmatrix} 11 & -2 & -8 & 0 \\ -2 & 17 & -7 & -5 \\ -8 & -7 & 21 & -6 \\ 0 & -5 & -6 & 15 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \\ 800 \end{bmatrix} \quad (104)$$

The results obtained are:
$V_1 = 93.45\text{V}, V_2 = 88.37\text{V}, V_3 = 100.16\text{V}, V_4 = 122.85\text{V}.$

The positive sign of the second order derivatives of functional $W_{elst}(V_1, V_2, V_3, V_4)$, in Equations (105) indicates the existence of a minimum value for the electrostatic energy $W_{elst}(V_1, V_2, V_3, V_4)_{\min} = 29.772$ mJ.

$$\frac{\partial^2 W_{elst}(V_1,V_2,V_3,V_4)}{\partial V_1^2} = 10^{-6} + 2\times10^{-6} + 8\times10^{-6} = \\ = 11\times10^{-6} > 0 \\ \frac{\partial^2 W_{elst}(V_1,V_2,V_3,V_4)}{\partial V_2^2} = 2\times10^{-6} + 3\times10^{-6} + 5\times10^{-6} + \\ +7\times10^{-6} = 17\times10^{-6} > 0 \\ \frac{\partial^2 W_{elst}(V_1,V_2,V_3,V_4)}{\partial V_3^2} = 6\times10^{-6} + 7\times10^{-6} + 8\times10^{-6} = \\ = 21\cdot10^{-6} > 0 \\ \frac{\partial^2 W_{elst}(V_1,V_2,V_3,V_4)}{\partial V_4^2} = 4\times10^{-6} + 5\times10^{-6} + 6\times10^{-6} = \\ = 15\times10^{-6} > 0 \quad (105)$$

Using the Nelder-Mead algorithm, with 490 iterations, the values of the electric potentials are:
$V_1 = 93.4026\text{V}, V_2 = 88.3149\text{V}, V_3 = 100.1083\text{V},$
$V_4 = 122.83344\text{V}.$

The minimum value of the electrostatic energy is $W_{elst\min} = 29.771$ mJ.

8.1.3. Discussion

From applications 1 and 2 we observed that the variational formulation is more computationally advantageous for circuits with many nodes (n>3) compared to the method based on Kirchhoff's laws. In the first case we obtain a system with 8 unknowns where for the second there are only 4 unknowns. Also, the potential values obtained by those methods are in good agreement.

8.2. Solution of a Resistor Circuit

The analysis of a resistor circuits is performed as the former applications.

8.2.1. Application 3

Consider a resistor circuit (DC) illustrated in Figure 11. For the numerical values of parameters of branch elements:
$E_1 = 20\text{V}, E_2 = 20\text{V}, R_1 = 2\Omega, R_2 = 8\Omega, R_3 = 4\Omega, R_4 = 10\Omega,$ $I_s = 6\text{A}, G_5 = 0.2\text{s}$, it is required to determine the voltages at the resistor terminals.

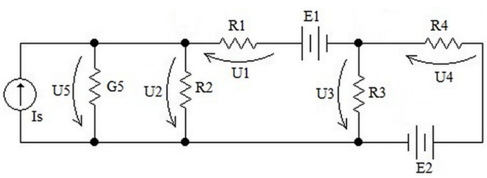

Figure 11. Resistor circuit with DC sources

If node voltages are used, for compare the two ways to solve the application, then all sources must be converted to voltage sources. Then, transfiguring the real current source, $I_s = 6\text{A}, G_5 = 0.2\text{s}$, into the real voltage source, $E_3 = 30\text{V}, R_5 = 5\Omega$, is obtained equivalent wiring diagram Figure 12.

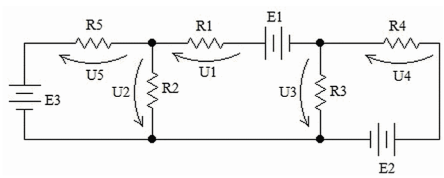

Figure 12. Resistor equivalent circuit with DC sources

Applying the solving algorithm with Kirchhoff's theorems, for stationary electrical circuits, we obtain the following linear system of equations, Equations (106) and (107):

$$\begin{bmatrix} \frac{U_1}{R_1} & 0 & 0 & \frac{U_4}{R_4} & \frac{U_5}{R_5} \\ -\frac{U_1}{R_1} & \frac{U_2}{R_2} & \frac{U_3}{R_3} & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_3 \\ E_1 \\ E_2 \end{bmatrix} \quad (106)$$

$$\begin{bmatrix} \frac{U_1}{2} & 0 & 0 & \frac{U_4}{10} & \frac{U_5}{5} \\ -\frac{U_1}{2} & \frac{U_2}{8} & \frac{U_3}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 30 \\ 20 \\ 15 \end{bmatrix} \quad (107)$$

with solutions:
$U_1 = -1.25\text{V}, U_2 = 7.08\text{V}, U_3 = 3.33\text{V}, U_4 = 22.08\text{V},$
$U_5 = -7.92\text{V}.$

Applying the variational solving method [18, 19], the potentials of the n-1 nodes of the circuit will be determined, the potential of the nth node being the reference. In accord with Figure 13, results:
$U_1 = V_1 - V_2 - E_1 = V_1 - V_2 - 20, U_2 = V_1 - E_2 = V_1 - 15,$
$U_3 = V_2, U_4 = V_1, U_5 = V_1 - E_3 = V_1 - 30.$

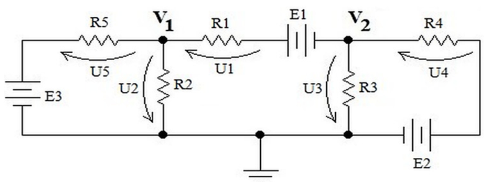

Figure 13. Resistor circuit with DC sources and unknowns' voltages V_1, V_2, V_3 and V_4

The functional based of the electrical power of the system is described by Equation (108):

$$P(V_1,V_2) = \frac{(V_1-30)^2}{5} + \frac{V_1^2}{10} + \frac{(V_1-V_2-20)^2}{2} + \\ + \frac{V_2^2}{4} + \frac{(V_2-15)^2}{8} \quad (108)$$

Differentiate $P(V_1,V_2)$ with respect to V_1, V_2 and set each result to zero, we found Equations (109), (110):

$$\frac{\partial P(V_1,V_2)}{\partial V_1} = 2\frac{(V_1-30)}{5} + 2\frac{V_1}{10} + 2\frac{(V_1-V_2-20)}{2} = 0 \\ \frac{\partial P(V_1,V_2)}{\partial V_2} = -2\frac{(V_1-V_2-20)}{2} + 2\frac{V_2}{4} + 2\frac{(V_2-15)}{8} = 0 \quad (109)$$

$$\begin{bmatrix} 8 & -5 \\ -4 & 7 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 160 \\ -65 \end{bmatrix} \quad (110)$$

The obtained results are: $V_1 = 22.08\text{V}, V_2 = 3.33\text{V}.$
From Equation (111) the minimum value of electrical power is obtained $P(V_1,V_2)_{\min} = 81.89\ \text{W}$:

$$\frac{\partial^2 P(V_1,V_2)}{\partial V_1^2} = \frac{2}{5} + \frac{2}{10} + \frac{2}{2} = 1.6 > 0 \\ \frac{\partial^2 P(V_1,V_2)}{\partial V_2^2} = \frac{2}{2} + \frac{2}{4} + \frac{2}{8} = 1.75 > 0 \quad (111)$$

Using the Nelder-Mead algorithm, for V_1 and V_2 varied between 0 and 30 volts, with 36 iterations, the potentials were $V_1 = 22.6\text{V}, V_2 = 2.91\text{V}$, with a minimum power value $P_{\min} = 82.6563\text{W}$, in good agreement with previous values. The graphical power is in Figure 14.

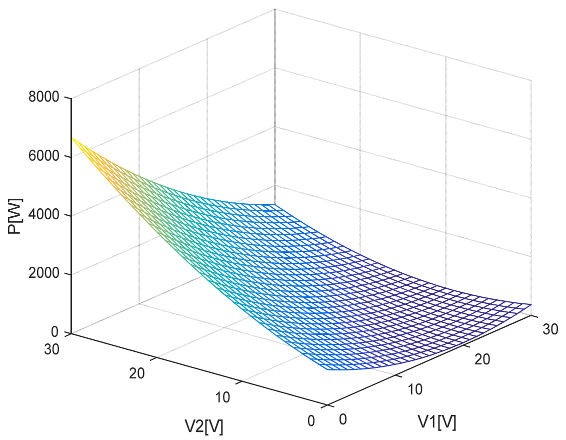

Figure 14. Power of electric circuit as a function of V_1 and V_2

9. DISCUSSION

Other conventional methods based of Kirchhoff's laws, such as nodal analysis (especially in this case), or mesh currents quickly give the same solutions, but the variational solution offer another approach, based on Hamilton variational principle, used throughout science and engineering. Is not necessary to consider the polarities of current/tensions to the branch of circuit, they are specified in the end, after obtaining the results.

10. CONCLUSIONS

This paper presents some fundamental aspects of calculus of variations as well as applications to electromagnetic field and electric circuits. Functionals associated with the PDEs for electromagnetic problems are presented. Algorithms of Rayleigh-Ritz and weighted residuals methods (Galerkin) are described and applied to electrostatic, magneto statics and electric circuits' problems. Variational principles can provide a useful alternative teaching method for solving capacitor and resistor circuits without the use of Kirchhoff's laws, using as a characteristic equation a functional that describes the entire circuit. The optimization procedures implemented in this method give the possibility to improve the numeric solutions and to increase the accuracy of the results. Future research will be focused on the nonlinear electric circuit's component.

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